# A comparison between linear programming model and optimal control model of production–inventory system

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**This study compares two models of the production–inventory system – optimal control and linear programming. We derived the optimality conditions of optimal control model and formulated the linear programming model. A new method to determine the theoretical solution of the boundary value problem has been suggested. Our numerical results suggest that control on the inventory level was realized at the end of the planning period, depending on the optimal control model, while in the linear programming model, it was realized from the beginning of the planning period. Also, the method to determine the theoretical solution of the boundary value problem has proven to be efficient.**

**Keywords:** Boundary value problem, deteriorating items, linear programming, optimal control, production–inventory system.

INVENTORY control plays a major role in production systems. Companies hedge the demand of products with the highest profit. Inventory control is one of the most important factors that contributes to the reduction of costs. It is therefore studied using continuous or periodic review policies. Model periodic-review inventory control strategies have gained widespread acceptance in the industry, and it is known that this model is an interesting alternative to the real-time control of industrial processes. This field is associated with problems, such as deterioration. The development of mathematical models serves to deal with the aforementioned problem, such as optimal control and linear programming.

An economic order quantity (EOQ) model with deteriorating inventory is one example. A model with constant rates of deterioration and production was developed by Khanra and Chaudhuri[1]. They assumed model with shortage, and the time of start and end production cycle is a random variable. Also with a constant rate of deterioration, Begum *et al.*[2] studied shortage, quadratic function of demand, and the effect of goods displayed, which represent inventory, on the sales. Jhaveri[3], and Karmakar and Choudhury[4] studied a model with holding costs as a function of time. Roy[5] and Bansal[6] developed an inventory system model with its deterioration rate as a function of time with and without shortages, and a demand rate as a function of price and constant respectively. The demand that rises quickly to a peak in the middle and falls quickly at the end of the planning time, i.e. a quadratic function, in a production–inventory system without shortage was studied by Gite[7]. A model that includes the cost of deterioration, with the deterioration rate as a random variable that follows a Weibull distribution, was developed by Sharma and Choudhury[8]. Further references on the EOQ model with deteriorating items can be found in the literature[9–16].

An economic production quantity (EPQ) model was used to determine the optimal production rate in inventory systems. Model with deterioration rate as a random variable that adheres to the Weibull distribution and discount price in a model of single items was addressed by Rao *et al.*[17], and Kawale and Bansode[18]. In the context of inflation, Pal *et al.*[19] clarified the length effect of the production cycle on the total cost with back orders, neglected lead time and loss in sales. Das *et al.*[20] discussed two constant rates of deterioration for raw materials and products for a single item model with constant rates of demand and holding costs.

An optimal control model of an inventory system was developed by Benhadid *et al.*[21] and Emamverdi *et al.*[22]. The former group assumed a linear deterioration function, single item, time functions of demand and holding cost, and two policies of inventory review – continuous and periodic. With periodic review policy, Emamverdi *et al.*[22] developed a model to realize administration goals on inventory levels and production rate with demand that depends on time.

Another mathematical model used for production–inventory system is linear programming. Most researchers intend to minimize total costs or maximize total profits. Zanoni and Zavanella[23], and Moengin and Fitriana[24] developed models for planning the production of steel. They took into account many parameters in their plan, such as storage space, multi-period, machine work, multi-product and production capacity. Lee and Kang[25] addressed a model with multi-period, only one order to

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each period, limited storage and without shortages with known time of replenishment. The target was to determine the quantity of replenishment for each period equal in duration. Linear and nonlinear models with single-stage, single-item, and machine-work hours were considered by Kefeli *et al.*[26]. They formulated a nonlinear clearing function and queuing system with a single server for the resource. Grimmett[27] developed a model with known inventory levels at the beginning and ending time periods, annual interest rates, and demand depending on the time. The aim of this work was to minimize the total cost of production and back orders. Veselovska[28] developed a model of production processes to induce more flexibility in production and minimize the total cost, which includes cost such as production, inventory and transportation. Talapatra *et al.*[29] analysed three cases of the workforce – fixed, change and both combined to reduce the production cost and meet the fluctuating demands by determining the levels of production, workforce and inventory.

The present model is useful in several ways. The first step involves the formulation of a linear programming model of the production–inventory system with deteriorating items. This is followed by the introduction of a new method to determine the theoretical solution of the boundary value problem. Then, we compare the results of the production–inventory system – optimal control and linear programming.

This article is organized in the following order. First we introduce the notations and assumptions involved in the optimal inventory model. Then we discuss the formulation of the optimal control model and derivation of the optimality conditions of the periodic-review system. The next section deals with the linear programming model followed by the section illustrating the results of the two aforementioned models. The final section summarizes our findings and suggests future researches.

## Notations and assumptions of the model

### *Notations*

The following variables and parameters were used:

$T$: length of the planning horizon ($T > 0$).
$Y(t)$: inventory level at time $t$.
$N(t)$: production rate at time $t$.
$D(t)$: demand rate for production at time $t$.
$\delta(t)$: deterioration rate, which depends on time.
$\hat{y}(t)$: inventory goal level.
$\hat{n}(t)$: production goal rate.
$Y(0)$: initial inventory level.
$h$: penalty incurred when the inventory level deviates from its goal level ($h > 0$).
$k$: penalty incurred when the total production rate deviates from its goal rate ($k > 0$).

### *Assumptions*

We took into account the following assumptions:

1. A firm can produce a certain product, sell some, and stack the rest in a warehouse.
2. Increasing demand rate.
3. The firm has set an inventory goal level and a production goal rate.
4. No shortage, and items are subjected to deterioration through storage.

The inventory management aims to control inventory at the specific level to reduce the cost and at the same time satisfy the exogenous demand without loss in sales, thus controlling the production quantity for sale and storage at a specific level. A specific level of inventory is possible at any time or at the end of the planning period depending on the inventory management.

## Optimal control model

The objective function can be expressed as the quadratic form to minimize eq. (1)[30]

$$2J = \sum_{t=0}^{T-1} h\{Y(t) - \hat{y}(t)\}^2 + k\{N(t) - \hat{n}(t)\}^2, \qquad (1)$$

subject to the state equation

$$\Delta Y(t) = N(t) - D(t);\ t = 0, 1, ..., t_1, \qquad (2)$$

$$\Delta Y(t) = N(t) - D(t) - \delta(t)Y(t);\ t = t_1 + 1, ..., T - 1, \qquad (3)$$

and positive constraint

$$N(t) > 0;\ t = 0, 1, ..., T - 1, \qquad (4)$$

with initial condition $Y(0) = y_0$, where $\Delta Y(t) = Y(t + 1) - Y(t)$ is called the difference operator.

### *Optimality conditions and solution of the model*

Inventory goal level and production goal rate must satisfy the state equations, i.e. eqs (2) and (3). Thus the production goal rate is given by

$$\hat{n}(t) = D(t);\ t = 0, 1, ..., t_1, \qquad (5)$$

$$\hat{n}(t) = D(t) + \delta(t)\hat{y}(t);\ t = t_1 + 1, ..., T - 1, \qquad (6)$$

The Lagrangian function is

$$L=\sum_{t=0}^{T-1}-\frac{1}{2}[h\{Y(t)-\hat{y}(t)\}^2+k\{N(t)-\hat{n}(t)\}^2]$$

$$+\sum_{t=0}^{t_1}\lambda(t+1)[N(t)-D(t)-Y(t+1)+Y(t)]$$

$$+\sum_{t=t_1+1}^{T-1}\lambda(t+1)[N(t)-D(t)-\delta'(t)Y(t)-Y(t+1)+Y(t)]$$

$$+\sum_{t=0}^{T-1}Z(t)N(t). \tag{7}$$

The Hamiltonian function is defined as

$$H(t)=-\frac{1}{2}[h\{Y(t)-\hat{y}(t)\}^2+k\{N(t)-\hat{n}(t)\}^2]$$
$$+\lambda(t+1)[N(t)-D(t)];\ t=0,1\ldots,t_1, \tag{8}$$

$$H(t)=-\frac{1}{2}[h\{Y(t)-\hat{y}(t)\}^2+k\{N(t)-\hat{n}(t)\}^2]$$
$$+\lambda(t+1)[N(t)-D(t)-\delta(t)Y(t)];$$
$$t=t_1+1,\ldots,T-1. \tag{9}$$

By using eqs (8) and (9), we can write eq. (7) as follows

$$L=\sum_{t=0}^{T-1}\{H(t)-\lambda(t+1)(Y(t+1)-Y(t))\}+\sum_{t=0}^{T-1}Z(t)N(t), \tag{10}$$

where $Z(t)$ is the Lagrange multiplier, which satisfies the complementary slackness conditions

$$Z(t)\geq 0;\ \ Z(t)N(t)=0. \tag{11}$$

From eqs (4) and (11), we get

$$Z(t)=0. \tag{12}$$

Equations (4), (8) and (9) are concave in $N(t)$. Thus eqs (8), (9) and (11) are the necessary and sufficient conditions for maximizing the Hamiltonian problem.

Now, differentiating eq. (10) with respect to $Y(t)$ yields

$$\Delta\lambda(t)=h\{Y(t)-\hat{y}(t)\};\ \ t=0,\ldots,t_1, \tag{13}$$

$$\Delta\lambda(t)=h\{Y(t)-\hat{y}(t)\}+\lambda(t+1)\delta(t);\ \ t=t_1+1,\ldots,T-1. \tag{14}$$

To get the terminal boundary conditions, we differentiate eq. (10) with respect to $X(T)$

$$\frac{\partial}{\partial X(T)}L=-\lambda(T)=0\rightarrow\lambda(T)=0. \tag{15}$$

To get the production rate, we differentiate eq. (10) with respect to $N(t)$

$$N(t)=\hat{n}(t)+\frac{1}{k}\lambda(t+1);\ t=0,1,\ldots,T-1. \tag{16}$$

Substituting eqs (5) and (16) into eq. (2) yields

$$\Delta Y(t)=\frac{1}{k}\lambda(t+1);\ t=0,1,\ldots,t_1. \tag{17}$$

Substituting eqs (6) and (16) into eq. (3) yields

$$\Delta Y(t)=-\delta(t)\{Y(t)-\hat{y}(t)\}+\frac{1}{k}\lambda(t+1);\ \ t=t_1+1,\ldots,T-1. \tag{18}$$

From eqs (13), (14), (17) and (18) we obtain the following system of difference equations

$$\left.\begin{array}{l}
\Delta Y(t)=\frac{1}{k}\lambda(t+1);\ \ t=0,1,\ldots,t_1,\\
\Delta Y(t)=-\delta(t)\{Y(t)-\hat{y}(t)\}+\frac{1}{k}\lambda(t+1);\\
\qquad t=t_1+1,\ldots,T-1,\\
\Delta\lambda(t)=h\{Y(t)-\hat{y}(t)\};\ t=1,\ldots,t_1,\\
\Delta\lambda(t)=h\{Y(t)-\hat{y}(t)\}+\lambda(t+1)\delta(t);\\
\qquad t=t_1+1,\ldots,T-1.
\end{array}\right\} \tag{19}$$

This boundary value problem can be solved numerically using Microsoft Excel with initial condition $Y(0)=y_0$ and the terminal condition $\lambda(T)=0$ (ref. 30).

### *Theoretical solution*

Benhadid *et al.*[21] and Emamverdi *et al.*[22] have used the sweep method to solve the boundary value problem. Here we propose a new method to solve eq. (19) as follows.

From eq. (19), we have

$$\Delta Y(t)=-\delta(t)\{Y(t)-\hat{y}(t)\}+\frac{1}{k}\lambda(t+1);\ \ t=0,\ldots,T-1, \tag{20}$$

$$\Delta\lambda(t) = h\{Y(t) - \hat{y}(t)\} + \lambda(t+1)\delta(t);\ \ t = 0,\ldots,T-1. \tag{21}$$

For $t = 0$, we get

$$Y(1) = Y(0) - \delta(0)\{Y(0) - \hat{y}\} + \frac{1}{k}\lambda(1),$$

$$Y(1) = \{1-\delta(0)\}Y(0) + \frac{1}{k}\lambda(1) + \delta(0)\hat{y}, \tag{22}$$

$$\lambda(1) = \frac{h}{1-\delta(0)}Y(0) + \frac{1}{1-\delta(0)}\lambda(0) - \frac{h}{1-\delta(0)}\hat{y}. \tag{23}$$

Substituting eq. (23) into eq. (22) yields

$$Y(1) = \{1-\delta(0)\}Y(0) + \frac{h}{k\{1-\delta(0)\}}Y(0)$$

$$+\frac{1}{k\{1-\delta(0)\}}\lambda(0) - \frac{h}{k\{1-\delta(0)\}}\hat{y} + \delta(0)\hat{y},$$

$$Y(1) = a_0Y(0) + b_0\lambda(0) + \{1-a_0\}\hat{y}, \tag{24}$$

where

$$a_0 = \{1-\delta(0)\} + \frac{h}{k\{1-\delta(0)\}},$$

$$b_0 = \frac{1}{k\{1-\delta(0)\}}. \tag{25}$$

Equation (23) becomes

$$\lambda(1) = e_0Y(0) + c_0\lambda(0) - e_0\hat{y}, \tag{26}$$

where

$$e_0 = \frac{h}{1-\delta(0)};\ \ c_0 = \frac{1}{1-\delta(0)}. \tag{27}$$

For $t = 1$, we get

$$Y(2) = \{1-\delta(1)\}Y(1) + \frac{1}{k}\lambda(2) + \delta(1)\hat{y}, \tag{28}$$

$$\lambda(2) = \frac{h}{1-\delta(1)}Y(1) + \frac{1}{1-\delta(1)}\lambda(1) - \frac{h}{1-\delta(1)}\hat{y}. \tag{29}$$

Substituting eq. (27) into eq. (29), we get

$$\lambda(2) = e_1Y(1) + c_1\lambda(1) - e_1\hat{y}. \tag{30}$$

Substituting eq. (29) into eq. (28) yields

$$Y(2) = \{1-\delta(1)\}Y(1) + \frac{h}{k\{1-\delta(1)\}}Y(1)$$

$$+\frac{1}{k\{1-\delta(1)\}}\lambda(1) - \frac{h}{k\{1-\delta(1)\}}\hat{y} + \delta(1)\hat{y}. \tag{31}$$

Substituting eq. (25) into eq. (31), we get

$$Y(2) = a_1Y(1) + b_1\lambda(1) + \{1-a_1\}\hat{y}. \tag{32}$$

Substituting eqs (24) and (26) into eq. (32) to get $Y(2)$ and with respect to $Y(0)$ and $\lambda(0)$ yields

$$Y(2) = \{(a_0 * a_1) + (e_0 * b_1)\}Y(0) + \{(b_0 * a_1)$$

$$+(c_0 * b_1)\}\lambda(0) + [\{(1-a_0) * a_1\}$$

$$-(e_0 * b_1) + (1-a_1)]\hat{y}. \tag{33}$$

We can write eq. (33) as

$$Y(2) = A(1)Y(0) + B(1)\lambda(0) + N(1)\hat{y}. \tag{34}$$

where

$$A(1) = (a_0 * a_1) + (e_0 * b_1),$$

$$B(1) = (b_0 * a_1) + (c_0 * b_1),$$

$$N(1) = \{(1-a_0) * a_1\} - (e_0 * b_1) + (1-a_1). \tag{35}$$

Substituting eqs (24) and (26) into eq. (30) yields

$$\lambda(2) = \{(a_0 * e_1) + (e_0 * c_1)\}Y(0)$$

$$+\{(b_0 * e_1) + (c_0 * c_1)\}\lambda(0)$$

$$+[\{(1-a_0) * e_1\} - (e_0 * c_1) - e_1]\,\hat{y}. \tag{36}$$

We can write eq. (36) as

$$\lambda(2) = E(1)Y(0) + C(1)\lambda(0) - E(1)\hat{y}, \tag{37}$$

where

$$E(1) = (a_0 * e_1) + (e_0 * c_1),$$

$$C(1) = (b_0 * e_1) + (c_0 * c_1). \tag{38}$$

For $t = 2$, we get

$$Y(3) = \{1-\delta(2)\}Y(2) + \frac{1}{k}\lambda(3) + \delta(2)\hat{y}, \tag{39}$$

$$\lambda(3)=\frac{h}{1-\delta(2)}Y(2)+\frac{1}{1-\delta(2)}\lambda(2)-\frac{h}{1-\delta(2)}\hat{y}. \tag{40}$$

Substituting eq. (27) into eq. (40), we get

$$\lambda(3)=e_2Y(1)+c_2\lambda(1)-e_2\hat{y}. \tag{41}$$

Substituting eq. (40) into eq. (39) yields

$$Y(3)=\{1-\delta(2)\}Y(2)+\frac{h}{k\{1-\delta(2)\}}Y(2)$$

$$+\frac{1}{k\{1-\delta(2)\}}\lambda(2)-\frac{h}{k\{1-\delta(2)\}}\hat{y}+\delta(2)\hat{y}. \tag{42}$$

Substituting eq. (25) into eq. (42), we get

$$Y(3)=a_2Y(2)+b_2\lambda(2)+\{1-a_2\}\hat{y}. \tag{43}$$

Substituting eqs (33) and (36) into eq. (43) to get $Y(3)$ with respect to $Y(0)$ and $\lambda(0)$ yields

$$\begin{aligned}Y(3)=\{&(a_0*a_1*a_2)+(e_0*b_1*a_2)\\&+(a_0*e_1*b_2)+(e_0*c_1*b_2)\}Y(0)\\&+\{(b_0*a_1*a_2)+(c_0*b_1*a_2)\\&+(b_0*e_1*b_2)+(c_0*c_1*b_2)\}\lambda(0)\\&+[\{(1-a_0)*a_1*a_2\}-(e_0*b_1*a_2)\\&+\{(1-a_1)*a_2\}+\{(1-a_0)*e_1*b_2\}\\&-(e_0*c_1*b_2)-(e_1*b_2)+(1-a_2)]\hat{y}.\end{aligned} \tag{44}$$

We can write eq. (44) as

$$Y(3)=A(2)Y(0)+B(2)\lambda(0)+N(2)\hat{y}, \tag{45}$$

where

$$\left.\begin{aligned}A(2)&=(a_0*a_1*a_2)+(e_0*b_1*a_2)\\&\quad+(a_0*e_1*b_2)+(e_0*c_1*b_2),\\B(2)&=(b_0*a_1*a_2)+(c_0*b_1*a_2)\\&\quad+(b_0*e_1*b_2)+(c_0*c_1*b_2),\\N(2)&=\{(1-a_0)*a_1*a_2\}-(e_0*b_1*a_2)\\&\quad+\{(1-a_1)*a_2\}+\{(1-a_0)*e_1*b_2\}\\&\quad-(e_0*c_1*b_2)-(e_1*b_2)+(1-a_2).\end{aligned}\right\} \tag{46}$$

Substituting eqs (33) and (36) into eq. (41) yields

$$\begin{aligned}\lambda(3)=\{&(a_0*a_1*e_2)+(e_0*b_1*e_2)\\&+(a_0*e_1*c_2)+(e_0*c_1*c_2)\}Y(0)\\&+\{(b_0*a_1*e_2)+(c_0*b_1*e_2)\\&+(b_0*e_1*c_2)+(c_0*c_1*c_2)\}\lambda(0)\\&+[\{(1-a_0)*a_1*e_2\}-(e_0*b_1*e_2)+\{(1-a_1)*e_2\}\\&+\{(1-a_0)*e_1*c_2\}-(e_0*c_1*c_2)\\&-(e_1*c_2)-e_2]\hat{y}.\end{aligned} \tag{47}$$

We can write eq. (47) as

$$\lambda(3)=E(2)Y(0)+C(2)\lambda(0)-E(2)\hat{y}, \tag{48}$$

where

$$\left.\begin{aligned}E(2)&=(a_0*a_1*e_2)+(e_0*b_1*e_2)\\&\quad+(a_0*e_1*c_2)+(e_0*c_1*c_2),\\C(2)&=(b_0*a_1*e_2)+(c_0*b_1*e_2)\\&\quad+(b_0*e_1*c_2)+(c_0*c_1*c_2).\end{aligned}\right\} \tag{49}$$

From eqs (37) and (48), we can write

$$\lambda(4)=E(3)Y(0)+C(3)\lambda(0)-E(3)\hat{y}. \tag{50}$$

In general

$$\lambda(T)=E(T-1)Y(0)+C(T-1)\lambda(0)-E(T-1)\hat{y}. \tag{51}$$

From eqs (26), (36) and (47), we can find $E(3)$ and $C(3)$ by making a network, where nodes represent values and arrows represent multiplication sign.

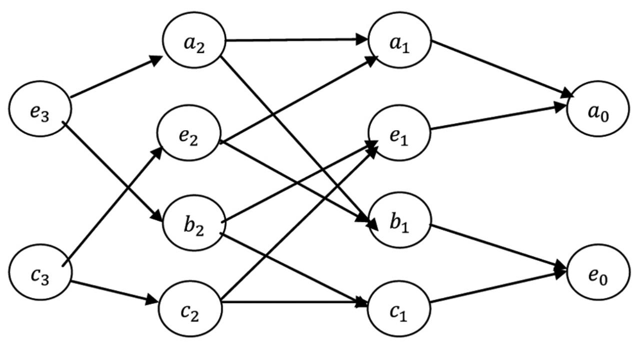


**Figure 1.** The network of $E(3)$.

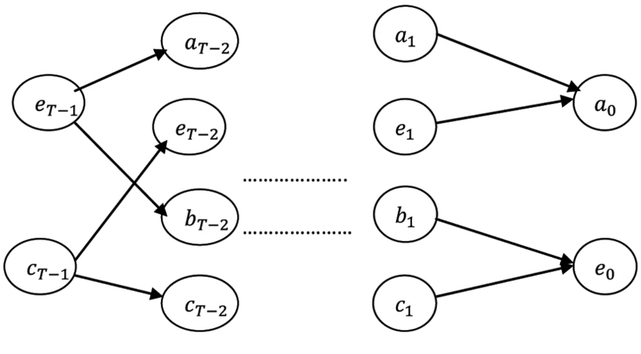


**Figure 2.** The network of $E(T-1)$.

For $E(3)$, the network is given by: $c_3$, $e_3$: the starting nodes; $a_0$, $e_0$: the ending nodes; $a$ and $e$: connect with $a$ and $b$; $b$ and $c$: connect with $c$ and $e$, where $e_3 \to a_2$ means $e_3 * a_2$ (Figure 1).

There are eight values from eight paths; the total of these values represents $E(3)$.

In general, the network of $E(T-1)$ is shown in Figure 2.

For $C(3)$, the network is similar to the network of $E(3)$; only the ending nodes are $b_0$ and $c_0$. i.e.

$$C(3) = \{e_3 \to a_0 + c_3 \to a_0\} * \left(\frac{b_0}{a_0}\right)$$

$$+\{e_3 \to e_0 + c_3 \to e_0\} * \left(\frac{c_0}{e_0}\right), \tag{52}$$

where $e_3 \to a_0$: all paths that start from $e_3$ end in $a_0$.

By applying the condition $\lambda(T) = 0$, we can find $\lambda(0)$, $\lambda(1)$ and $y(1)$ from eqs (51), (23) and (22) respectively.

## Formulation of the linear programming model

The objective function can be written as

$$\left.\begin{aligned} &\text{Min } J = h\{\hat{y} - Y(0)\} + \sum_{t=1}^{T} h\{Y(t) - \hat{y}\} \\ &\qquad + \sum_{t=0}^{T} k\{N(t) - \hat{n}(t)\};\ \hat{y} > Y(0), \\ &\text{Min } J = h\{Y(0) - \hat{y}\} + \sum_{t=1}^{T} h\{\hat{y} - Y(t)\} \\ &\qquad + \sum_{t=0}^{T} k\{N(t) - \hat{n}(t)\};\ \hat{y} < Y(0). \end{aligned}\right\} \tag{53}$$

Inventory level constraints are

$$Y(t+1) - Y(t) - N(t) + D(t) = 0;\ t = 0, 1, \ldots, t_1, \tag{54}$$

$$Y(t+1) - N(t) + D(t) + \{1 - \delta(t)\}Y(t) = 0;$$

$$t = t_1 + 1, \ldots, T - 1. \tag{55}$$

Inventory goal level constraints are

$$\hat{y} - 50 = 0, \tag{56}$$

$$\left.\begin{aligned} &Y(t) - \hat{y} \geq 0;\ t = 1, 2, \ldots, T;\ \hat{y} > Y(0), \\ &\hat{y} - Y(t) \geq 0;\ t = 1, 2, \ldots, T;\ \hat{y} < Y(0). \end{aligned}\right\} \tag{57}$$

The initial inventory level constraint is

$$Y(0) = 0. \tag{58}$$

Equations (56) and (58) assume that $Y(0) = 0$ and $\hat{y} = 50$.

Production goal rate constraints are

$$\hat{n}(t) - D(t) = 0;\ t = 0, 1, \ldots, t_1, \tag{59}$$

$$\hat{n}(t) - D(t) - \delta(t)\hat{y} = 0;\ t = t_1 + 1, \ldots, T - 1. \tag{60}$$

Production rate constraints are

$$\left.\begin{aligned} &N(t) - \hat{n}(t) \geq 0;\ t = 0, 1, \ldots, T - 1 \\ &N(t) > 0;\ t = 0, 1, \ldots, T - 1). \end{aligned}\right\} \tag{61}$$

Deterioration constraint is

$$\delta(t) - 0.05t = 0;\ t = t_1 + 1, \ldots, T - 1. \tag{62}$$

Demand constraint is

$$D(t) - 150 - 5t = 0;\ t = 0, 1, \ldots, T - 1. \tag{63}$$

## Numerical solution

Consider an inventory system with the following parameter values in proper units: $\hat{y} = 50$ items; $y_0 = 0$ items; $T = 6$ months; $t_1 = 2$ months; $k =$ US\$ 30; $h =$ US\$ 20; $D(t) = 150 + 5t$

$$\delta(t) = \begin{cases} 0; & t = 0, 1, 2, \\ 0.05t; & t = 3, 4, 5, 6. \end{cases}$$

### *Solution of the optimal control model*

Using the goal seek function in Microsoft Excel, we find the solution of the system eq. (19).

The simulation results (Figure 3) show that the optimal inventory level is converging to its goal level, as desired. In the first two months there is no deterioration in inventory, then it increases over time. Figure 4 shows that the optimal production rate, with an increasing demand, converges to its goal over time.

We solve eq. (19) using the present method as follows.

From eq. (51), we have

$$\lambda(6) = E(5)Y(0) + C(5)\lambda(0) - E(5)\hat{y}.$$

The network of $E(5)$ is shown in Figure 5.

$$E(5) = 1655.105 + 549.032 = 2204.137.$$

From eq. (52), we have

$$C(5) = \{e_5 \to a_0 + c_5 \to a_0\} * \left(\frac{b_0}{a_0}\right)$$

$$+\{e_5 \to e_0 + c_5 \to e_0\} * \left(\frac{c_0}{e_0}\right),$$

$$C(5) = 1655.105 * \left(\frac{0.033333}{1.666667}\right)$$

$$+549.032 * \left(\frac{1}{20}\right) = 60.554.$$

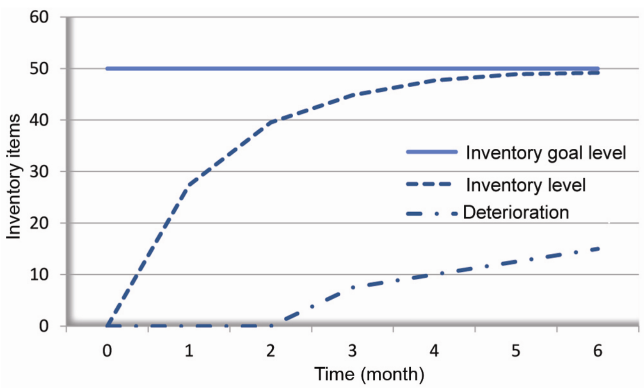


**Figure 3.** The inventory level according to the optimal control model.

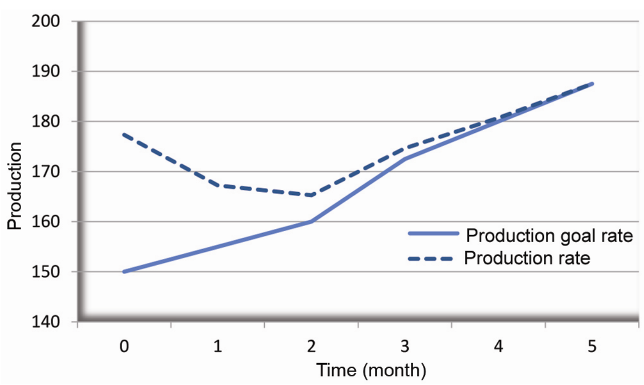


**Figure 4.** Production rate according to the optimal control model.

By applying the condition $\lambda(T) = 9$, we can find $\lambda(0)$ as

$$0 = 2204.137 * 0 + 609.554 * \lambda(0) - 2204.137 * 50,$$

$$\lambda(0) = 1819.985.$$

From eqs (6), (16), (22) and (23), we can find the inventory level and production rate.

The results in Table 1 are similar to those found using Microsoft Excel. Therefore, the present method is efficient to find a solution to the boundary value problem.

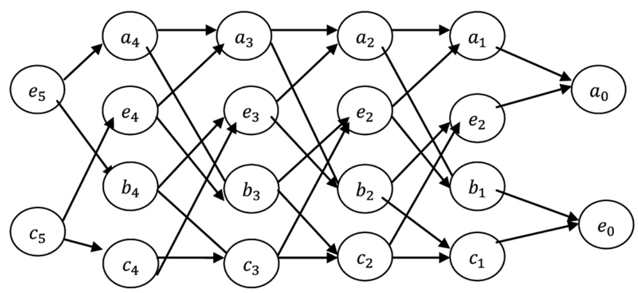


**Figure 5.** The network of $E(5)$.

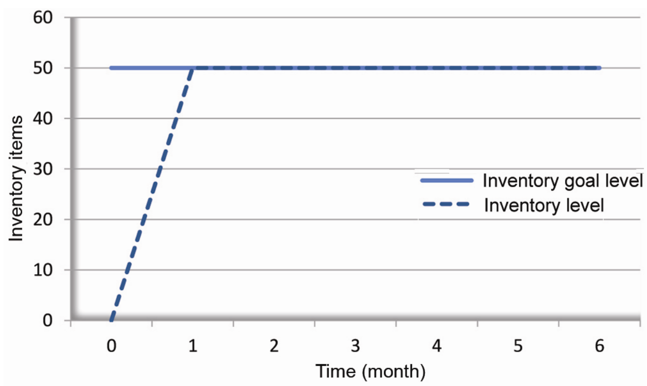


**Figure 6.** The inventory level according to the linear programming model.

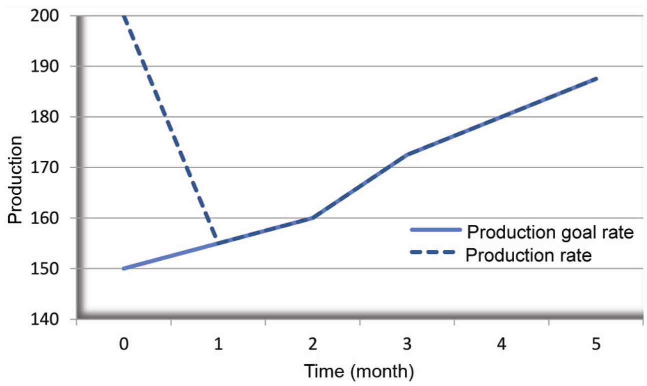


**Figure 7.** The production rate, according to the linear programming model.

**Table 1.** Solution of the production–inventory system

| | Time (month) | | | | | | |
|---|---|---|---|---|---|---|---|
| Model variables | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $Y$ | 0.0 | 27.3 | 39.6 | 44.8 | 47.7 | 48.9 | 49.2 |
| $\hat{n}$ | 150 | 155 | 160 | 172.5 | 180 | 187.5 | |
| $N$ | 177.3 | 167.2 | 165.3 | 174.6 | 180.7 | 187.5 | |
| $\delta\hat{y}$ | 0 | 0 | 0 | 7.5 | 10 | 12.5 | 15 |
| $\lambda$ | 1819.985 | 819.985 | 366.642 | 157.728 | 63.488 | 22.019 | 0.000 |

**Table 2.** Solution of the linear programming model.

| | Time (month) | | | | | | |
|---|---|---|---|---|---|---|---|
| Model variables | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $Y$ | 0.0 | 50 | 50 | 50 | 50 | 50 | 50 |
| $\hat{n}$ | 150 | 155 | 160 | 172.5 | 180 | 187.5 | |
| $N$ | 200 | 155 | 160 | 172.5 | 180 | 187.5 | |
| $\delta\hat{y}$ | 0 | 0 | 0 | 7.5 | 10 | 12.5 | 15 |

The production rate decreases in the first two months and then increases over time to compensate the deterioration in inventory.

### *Solution of the linear programming model*

Using the MatLab software (version 8.5), we obtain the following results.

The simulation results (Figure 6) show the optimal inventory level up to its goal level from the beginning of planning period. Figure 7 shows the optimal production rate, with an increasing demand, up to its goal.

The following can be deduced from Tables 1 and 2:

1. Optimality realized at the end of the planning period, according to the optimal control model.
2. Optimality realized at the beginning of the planning period, according to the linear programming model.
3. There is similarity in the deterioration in the two models.

## Conclusion and recommendations

In this study, we have developed two models of production–inventory system – optimal control and linear programming – to achieve the administration goals in inventory level and hedge demand.

The results are similar to those reported using Microsoft Excel, proving the efficiency of the method in solving boundary value problems. Moreover optimality was achieved at the end of the planning period in the case of the optimal control model, while in the case of the linear programming model, it was achieved at the beginning of the planning period. Therefore, optimal control model was found suitable in case of the administration is planned to the inventory level at the end of the planning period, while maintaining the inventory level from the beginning to the end of planning period was achieved by linear programming. Economically, these models were found to be efficient for inventory control, with deteriorating items. This study could be extended to include the stochastic demand or holding cost as a function with and without shortage.

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