M. S. Narasimhan

T. R. Ramadas

TIFR

M. S. Narasimhan is a towering figure in Indian science. He has international reputation as an extraordinarily versatile and deep mathematician, successful teacher and an efficient and correct administrator. It was my good fortune to have been his student. In this article, I have tried to give an account of his life, personality, and above all, his work. I have made special effort to give a sense of his major mathematical achievements.

Narasimhan himself gives a vivid account of his career and the major influences on him in an interview with R. Sujatha (http://www.asiapacific-math-news.com/03/0302/0025_0030.pdf). Two of his close collaborators have written essays about the man and his mathematics. The first of these is by C. S. Seshadri in *The Collected Papers of M.S. Narasimhan* (ed. Nitsure, N.), Hindustan Books Agency, New Delhi, 2007. The second, a warm tribute by S. Ramanan to his teacher, is available as http://www.asia-pacific-mathnews.com/03/0302/0021_0024.pdf.

In Appendix 1, I give an informal introduction to the mathematics that we will encounter. The reader can browse through this or refer to it as I describe the themes that run through Narasimhan's work. We will repeatedly encounter a type of construct that is typical of modern mathematics – sets with structure.

I close this introduction highlighting two insights in particular that have proved to be of huge import. The first is that while classifying a class of algebro-geometric objects, the *semistable* ones

have to be identified first; then the whole family can be built, lego-style, starting with a 'dominant' part that parametrizes these semistable objects, and then adding the other 'lego-pieces', each built out of families of 'smaller' semistable objects. Each semistable object is an 'extension' of *stable* objects. The second insight is that stable objects are characterized as those that satisfy nonlinear (partial differential) equations. Chronologically, these came in reverse order. The second has its origins in the work of Narasimhan and Seshadri, while the first, sprang from the work of Harder and Narasimhan (both works are dealt with later in the text).

A brief biography

Narasimhan was born on 7 June 1932 in the small village of Tandarai, in what is now Tamil Nadu. The nearest school was 5 miles away, and he used to drive a bullock cart to class. One day he came out at lunch time to feed the bullocks as usual, and found that they had disappeared, having managed to free themselves from their tether and wandered back home.

I quote (with some small edits) from an article I wrote for *The Mathematics Student:* 'From Tandarai Narasimhan went to Loyola College in Madras, where Father Racine (a Jesuit priest, French, and himself a student of Elie Cartan) noticed the young man's talents and advised him to go to the Tata Institute of Fundamental Research (TIFR) in Bombay, where a School of Mathematics had just been founded by the visionary K. Chandrasekharan. He was accompanied by fellow-student C. S. Seshadri. Chandrasekaran, an analyst and number theorist, was well aware of the winds of change blowing from France and elsewhere. He encouraged his protegés to learn the most modern and powerful mathematics, and facilitated this by arranging for a string of outstanding people – Warren Ambrose, Samuel Eilenberg, Laurent Schwartz and others – to lecture at TIFR. By all accounts, the combination of extraordinary teachers and brilliant students produced conditions that are seldom repeatable. After a little hesi-

tation, Narasimhan established himself as among the most original in the group, and went on to become one of India's most distinguished mathematicians.'

Here is the chronology: Narasimhan joined TIFR in 1953, spent the years 1957–1960 as a CNRS Research Associate in Paris, got his Ph D in 1960 from Bombay University, was made Professor at TIFR in 1965, and retired from TIFR as Professor of Eminence in 1992. Then he moved to the International Centre for Theoretical Physics (ICTP) at the invitation of Abdus Salam, to lead the mathematics group there during 1993–1999. He returned to India in 2003, after a further three years at the International School for Advanced Studies (SISSA), Trieste. He now lives in Bangalore and is a Distinguished Associate at the Indian Institute of Science (IISc), and Honorary Fellow of TIFR.

Narasimhan is a Fellow of the Indian Academy of Sciences, Bangalore; the Indian National Science Academy, New Delhi and the Royal Society (London) and a Chevalier de l'ordre National du Mérite (France). He is a recipient of the Shanti Swarup Bhatnagar Prize, the Third World Academy Award for Mathematics, the Padma Bhushan and the King Faisal International Prize for Science. He was the Founder-President of the National Board for Higher Mathematics in India. He has been President of the International Mathematical Union's (IMU's) Commission on Development and Exchange, Vice-President of the International Center of Pure and Applied Mathematics in France and member of the Executive Committee (EC) of IMU.

Narasimhan is married to Sakuntala Narasimhan, a musician equally accomplished in the Carnatic and Hindustani traditions, journalist and consumer advocate. They have two children. Mohan is a management professional based in Boston, and his older sister Shobhana is a physicist, Professor at the Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore.

To conclude this biography, let me attempt a reconstruction of the influences that shaped Narasimhan's mathematical personality. Father Racine, certainly. Then

Warren Ambrose's lectures at TIFR – Narasimhan recalls how Ambrose introduced the rudiments of general topology in two lectures and then quickly went on to functional analysis and the Peter–Weyl theorem. A favourite quote: 'Cauchy's theorem [in complex function theory] is not a statement about holomorphic functions, but one about the integral of a closed form.' S. Eilenberg taught algebraic topology; 'without ever mentioning the words categories or functors, [he] taught the whole course from a functorial viewpoint'. Another major influence was Laurent Schwartz, who instilled a lifelong fascination with partial differential equations. It is also a tribute to the broad culture of Schwartz that Narasimhan's initiation into geometry was as a note-taker for Schwartz's lectures on complex manifolds. Narasimhan credits long conversations with K. Chandrasekharan and K. G. Ramanathan for his love of number theory and algebraic groups.

During a long spell in hospital in France, Narasimhan read through a preliminary version of a foundational paper by Kodaira and Spencer on deformations of complex structures, and this laid the foundation for much of his later work.

Fractional powers of elliptic operators

While in Paris, Narasimhan collaborated with a young Japanese mathematician, T. Kotake, on his first major work. For those who know Narasimhan only as a geometer, a glance at this paper will be particularly striking.

For suitable domains and boundary conditions, the Laplacian, defined as a partial differential operator on smooth functions, has extensions as an unbounded, self-adjoint, positive operator on square-integrable functions. Consider such a positive self-adjoint realization of an elliptic operator in L^2. For such an operator A and any complex number s,

one can define A^s using the spectral decomposition. Kotake and Narasimhan first proved that the kernels of these 'fractional' powers are 'very regular'. This result was used in the original proof of the Atiyah–Bott fixed-point theorem.

In the further study of the real analyticity properties of these kernels, when the coefficients of the operator are analytic, the key was a remarkable theorem which guarantees that a function u is analytic, if and only if on every compact set the L^2-norms of $A^k u$, $k \in N$ satisfy Cauchy-type inequalities. This contains the well-known result that given an elliptic differential operator with analytic coefficients, its solutions are also analytic.

The universal connection

When Narasimhan returned to Bombay, he met Ramanan, who had joined TIFR as a graduate student. Together, they proved a remarkable theorem in differential geometry, which I will now describe. The theorem applies to arbitrary principal bundles, but in this exposition I consider bundles with unitary structure group. In this case, there is an equivalent statement for complex unitary vector bundles, and this is what we will consider.

To set the stage, I need to introduce the *universal bundle*. Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space. Fix a natural number r and consider the 'grassmannian' $G_r(\mathcal{H})$ of r-dimensional subspaces of $\mathcal{H}$. This is an infinite-dimensional manifold, each of whose points represents a r-dimensional subspace. (We can in fact consider the corresponding grassmannian in a finite-dimensional Hilbert space of 'large enough' dimension, which is what Narasimhan and Ramanan did. The infinite-dimensional formulation leads to slightly more felicitous formulations.)

There is a natural, 'tautological', complex, unitary, rank r vector bundle $\mathcal{E}_r$ on $G_r(\mathcal{H})$. Namely, one associates to each point of $G_r(\mathcal{H})$, the subspace of $\mathcal{H}$ that it represents. The bundle $\mathcal{E}_r$ is 'universal' in the following sense: given *any* complex, unitary, rank r vector bundle $E \to X$, there exists a map $\phi: X \to G_r(\mathcal{H})$, such that E is isomorphic to the 'pull-back' bundle $\phi^*\mathcal{E}_r$. In other words, there is a bundle map $\tilde{\phi}: E \to \mathcal{E}_r$ preserving metrics. Note that the map ϕ is far from unique; in fact, homotopic maps induce isomorphic bundles.

This much is topology. Differential geometry comes into the picture with the following observation. The bundle $\mathcal{E}_r \to G_r(\mathcal{H})$ carries a unique connection invariant under the unitary group $U(\mathcal{H})$. The theorem on universal connections mentions in this context that *given any unitary connection ∇ on E, there is a bundle map $\tilde{\phi}: E \to \mathcal{E}_r$ compatible with connections*.

The crux of the proof is a local argument which is illuminating already in the simplest possible case, that of a trivial line bundle on a disc; see the lemma in Appendix 2. I include this because (a) every mathematical exposition should contain a proof; (b) this one involves elementary calculus but is still not obvious, and (c) it illustrates one of Narasimhan's strengths, finding the 'toy' example which holds the key to a problem.

The existence of universal connections has been used extensively – in the work of Chern–Simons, in the definition of the Cheeger–Simons differential character, in work on super-connections and in the context of stochastic differential equations.

Stable and unitary bundles

Mathematical theorems are rarely described as *discoveries*. When they were barely 30 years old, Narasimhan and Seshadri made a remarkable discovery at the crossroads of the algebraic and complex analytic geometry of the era. Inspired by close readings of the works of Kodaira–Spencer and André Weil, Narasimhan and Seshadri began an investigation of the space of irreducible unitary vector bundles of given rank r on a Riemann surface of genus g. They proved that this space has a natural structure of a complex manifold of dimension $r^2(g-1) + 1$.

Fortune favours the brave. Here is how the thrilling tale then unfolded:

- David Mumford, who was engaged in combining the older ideas of Hilbert concerning invariant theory with modern foundational work of Grothendieck in algebraic geometry, introduced the algebra-geometric notion of a *stable* vector bundle and announced the result that the space of stable bundles (of rank r on a curve of genus g) is naturally an algebraic variety of dimension $r^2(g-1) + 1$.
- This definition and result suggested to Narasimhan and Seshadri that

TIFR

irreducible unitary bundles and stable bundles (of degree zero) were essentially the same objects (the term 'degree' is explained in Appendix 1).

- The direction *irreducible and unitary* $\Rightarrow$ *stable of degree zero* was not difficult.
- In a *tour de force*, they proved the converse direction using a strategy that goes back to Klein and Poincaré, called the *continuity method*.

The idea of the continuity method is the following. Since irreducible unitary bundles are stable, there is a map from the moduli space of irreducible unitary representations to the (connected) moduli space of stable bundles (of degree zero). An infinitesimal computation shows that (this map is complex–analytic and) the differential is an isomorphism, so the map is open. The heart of the proof is to show that the map is closed, which shows its surjectivity.

Since the definition of stability is so central to what follows, I will deal with this notion here.

Definition 1. A holomorphic vector bundle E on a smooth Riemann surface is said to be *semistable* if for every proper holomorphic sub-bundle $0 \neq F \subset E$, we have

$$\frac{\text{degree } F}{\text{rank } F} \leq \frac{\text{degree } E}{\text{rank } E}.$$

If strict inequality holds, we say that E is *stable*.

The best way to motivate this definition is to prove the 'easy' implication above as an exercise in complex differential geometry, the key calculation being one that uses the Chern–Weil definition of degree in terms of curvature, and the principle that 'curvature decreases in sub-bundles'. Two important algebro-geometric consequences flow from this topological constraint. Namely, semistable bundles form a bounded family, and this family, modulo isomorphism (strictly speaking, '*s*-equivalence') can be embedded in some projective space by a natural family of 'theta-functions'. All this follows from Mumford's geometric invariant theory and the work of Seshadri.

The moduli spaces $U_X(r, d)$

Riemann surfaces of a fixed genus g are parametrized by an algebraic variety of dimension $\mathcal{M}_g$. The study of these parameter spaces goes back at least to Riemann. Local coordinates (i.e. parameters) are traditionally called 'moduli', and the space itself called a space of moduli, or simply, a 'moduli space'.

The theorem of Narasimhan and Seshadri focused attention on a new kind of moduli space associated to a *fixed* Riemann surface X. Fix two integers r and d, with r required to be positive. Let $U_X^{\text{st}}(r, d)$ denote the set of isomorphism classes of stable bundles E on X with rank $E = r$ and degree $E = d$. This is a smooth algebraic variety which is not compact unless the integers r and d are co-prime (i.e. without common divisors) – note that in this case any semistable bundle is guaranteed to be stable. If r and d *have* common divisors, one can compactify $U_X^{\text{st}}(r, d)$ by adding points that correspond to '*s*-equivalence classes of (non-stable but) semistable vector bundles'. This construction is due to Seshadri. The resulting compact varieties are denoted $U_X(r, d)$.

A systematic study of the spaces $U_X(r, d)$ was begun by Narasimhan and Ramanan. Over a period of 10 years, in what Narasimhan has described as his most intense collaboration, they explored the geography of these moduli spaces and proved a number of basic facts about them. In the process they uncovered connections with classical geometry and proved a 50-year-old conjecture. Their results and techniques (notably a technique they called the Hecke transform) have been revisited by others, notably in arithmetic geometry.

The Harder–Narasimhan filtration

We come next to a richly textured chapter in modern algebraic geometry. Soon after the theorem of Narasimhan and Seshadri was proved, P. Newstead used it to compute the Betti numbers (i.e. the dimensions of the cohomology groups) of $U_X(2, d)$. G. Harder in turn, used the information to verify the Weil conjectures (cf. Appendix 1) for this variety. Narasimhan and Harder then turned the mathematics on its head, used the Weil conjectures to extend Newstead's computations to higher rank. In the process, they discovered an important stratification of the set (in fact, in modern language, the *stack*) of all vector bundles. A 'stratification' is a partition – division into disjoint subsets – with special geometric properties.

The integers r, d will be taken to be co-prime. I will now describe the strategy of the computation, postponing the description of the strata to the end of this section.

In what follows, one has to work with moduli spaces of bundles with 'fixed determinant'. That is, one has to fix a holomorphic line bundle L of the relevant degree, and restrict oneself to vector bundles E such that $\det E \sim L$. We will let $U_X(r, L)$ denote the moduli space of rank r bundles E with determinant L.

- The Betti numbers of $U_X(r, L)$ depend only on the genus of X, rank r and degree d, so we can assume that the curve X and the line bundle L are defined over $\mathbb{Q}$, and then this will be true for the moduli space $U_X(r, L)$ as well.
- The variety $U_X(r, L)$ is smooth and compact – it is here that we use the fact that r and d have no common divisors – so the Weil conjectures apply. In other words, to determine the Betti numbers of the nonsingular complex variety $U_X(r, L)$, it suffices to consider the corresponding variety
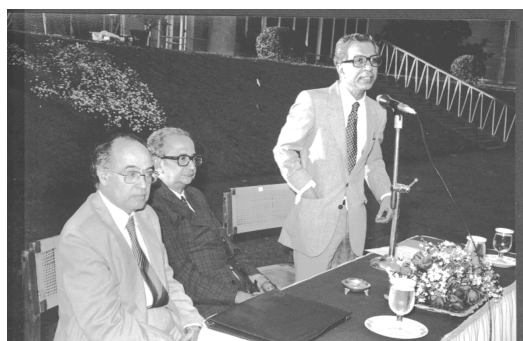
International Conference on Vector Bundles on Algebraic Varieties, TIFR, 1984. From left to right: M. F. Atiyah, B. V. Sreekantan and M. S. Narasimhan.

'over $\mathbb{F}_q$' for some suitable family of finite fields and 'count the $\mathbb{F}_q$-rational points'. (When an algebraic variety is defined by equations with integer coefficients, and slightly more generally, a solution in a finite field $\mathbb{F}_q$, with q a power of a prime p, is called a $\mathbb{F}_q$-rational point. If $q = p$, these are just solutions modulo p.) Equivalently, one needs to count the number $N_q(r, d)$ (isomorphism classes) of stable vector bundles of rank r and degree d with fixed determinant L. The requirement that the bundles be stable, and be defined over $\mathbb{F}_q$, makes this number finite.

- A classical result of Siegel (regarding the 'Tamagawa number' of the group $SL(n)$) yields a formula (the right-hand side of which we simply denote R) for the sum:

$$\sum_E \frac{1}{|\text{Aut } E|} = R.$$

Here the sum is infinite and ranges over the set of isomorphism classes of *all* vector bundles – stable or otherwise – defined over $\mathbb{F}_q$ such that det $E \sim L$. What makes the sum finite is the weight factor $\frac{1}{|\text{Aut } E|}$, where Aut E is the group of automorphisms ('symmetries') of the bundle E and $|\text{Aut } E|$ denotes its cardinality.

- We write the last formula as a sum as a contribution from the finite set of stable bundles, and the rest. Noting that the group of automorphisms of a stable bundle is just the multiplicative group of nonzero elements of $\mathbb{F}_q$, we get

$$R = \sum_{E \text{ stable}} \frac{1}{|\text{Aut } E|} + \sum_{E \text{ not stable}} \frac{1}{|\text{Aut } E|}$$

$$= \frac{1}{q-1} N_q(r, d) + \sum_{E \text{ not stable}} \frac{1}{|\text{Aut } E|}.$$

(The notation $N_q(r, d)$ is explained in the second bullet-point.) Rewriting, we get

$$N_q(r, d) = (q - 1)$$
$$\left\{ R - \sum_{E \text{ not stable}} \frac{1}{|\text{Aut } E|} \right\}.$$

All the sums are over bundles E with determinant isomorphic to L.

- This is where the Harder–Narasimhan stratification enters the picture. The 'biggest' stratum consists of semistable bundles. Since r and d are assumed co-prime, these are all stable, and the count of these gives $N_q(r, d)$. The other strata partition the set of non-stable bundles. This allows us to compute the term $\sum_{E \text{ not stable}} (1/|\text{Aut } E|)$ inductively in terms of $N_q(r', d')$ for ranks r' strictly smaller than r.

The stratification has its origins in the *Harder–Narasimhan* (*H–N*) *filtration of a non-semistable bundle*, which I now explain. Consider first a rank 2 bundle E. If E is not semistable, by definition there exists a line subbundle $L \subset E$ with 2 degree $L >$ degree E. Among all such, there is a unique L with the maximum degree, say d. This yields a canonical filtration $0 \subset L \subset E$. More generally, for a non-semistable E of arbitrary rank, there is a canonical filtration by sub-bundles

$$0 \subset E_1 \subset E_2 \cdots \subset E_{m-1} \subset E_m = E,$$

with the successive quotients being semistable and with decreasing slopes, $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E/E_{m-1})$. The *slope $\mu(E)$* of a bundle E is

$$\mu(E) = \frac{\text{degree } E}{\text{rank } E}.$$

The Harder–Narasimhan strata are defined by grouping together into one stratum all E with a given type of H–N filtration, the 'type' being the sequence of slopes and ranks associated with the H–N filtration.

The H–N filtration has been extremely useful in many other contexts in algebraic geometry and number theory.

Representation theory

During a year-long (1968–69) visit to the Institute for Advanced Study, Princeton, Narasimhan made foray into the representation theory of real Lie groups, in joint work with K. Okamoto.

The most natural linear representations of a group are obtained by considering the action of the group on a set and then the associated action on (real or complex-valued) functions on the set. Often there are natural subspaces of functions invariant under the action. For compact

Lie groups, the Borel–Weil theorem realizes finite-dimensional irreducible representations on the space of holomorphic sections of homogeneous line bundles on flag varieties. Physicists are familiar with the action of the Poincaré group on solutions of the wave equation.

A general conjecture of R. P. Langlands regarding the realization of discrete series of Harish Chandra – which are certain important representations of a real noncompact Lie group G – predicted, in the hermitian symmetric case, that these representations would be realized on square integrable cohomology of certain holomorphic vector bundles on G/K. (Here K is the maximal compact subgroup of G and the 'hermitian symmetric case' is when G/K admits an invariant complex structure.) This is altogether subtler than the Borel–Weil theorem, in that these spaces are non-compact and the representations infinite-dimensional and serious questions of analysis come into play.

The work of Narasimhan and Okamoto was the most significant step towards the proof of Langlands' conjecture in general.

In the case of symmetric spaces which are not hermitian symmetric, a substitute had to be found for the Cauchy–Riemann equations which characterize holomorphic sections. It was Narasimhan's idea to replace the Cauchy–Riemann equations with the Dirac equation. (Rather modestly, he says 'it was in the air'.) This he suggested to his student R. Parthasarathy. The resulting series of works were major landmarks in representation theory.

Mathematical physics

Mathematics and theoretical physics have always had a close relationship, though each field has its own 'big questions' and working style. (Narasimhan likes to emphasize that much of pure mathematics has its own internal dynamics.)

In the seventies, there began a particularly fruitful period of interaction between the two disciplines. This began with the realization that a (classical) gauge field – for example, the electromagnetic potential $A_\mu(x)$ – can and should be regarded as a connection one-form. Artefacts like the 'Dirac string' that enters the description of a magnetic monopole disappear once it is realized that connections on nontrivial bundles provide a natural description. The study

of solitons and instantons required further inputs from topology, and then came the discovery that algebraic and differential geometry can be deployed to understand the solutions of the corresponding nonlinear partial differential equations. As physicists became familiar with the geometry, they incorporated relatively exotic geometric structures, for example, from the work of Chern and Simons. (This work uses crucially universal connections.) Atiyah and Bott, investigating 'toy models' of Yang–Mills theory, discovered bridges between infinite-dimensional Morse theory and the theorem of Narasimhan and Seshadri, as well as the computations of Harder and Narasimhan. Conformal field theory proved to involve representations of infinite-dimensional Lie algebras and produced intriguing conjectures about 'linear systems' on moduli spaces of vector bundles. In the past two decades, string theory has generated wonderful conjectures relating enumerative geometry and modular forms.

Narasimhan was not indifferent to these developments. I joined the Tata Institute in 1977 as a research student in theoretical physics. P. P. Divakaran, my advisor, was one of Narasimhan's friends and interlocutors, and persuaded him to give a series of lectures on vector bundles, connections and characteristic classes. I was asked to be note-taker, and this proved to be a turning point in my career. I was a callow physics student, with a good training from IIT Kanpur and a taste for the formal. Over the next two years, I reported to Narasimhan my attempts to understand gauge theories (including the problems of gauge-fixing as revealed by the Gribov ambiguity) and Dirac's theory of constrained systems. Then I watched in awe as he laid bare the geometry underlying the theory, in work that became the body of my thesis. First came the formal part, a succinct formulation of the theory of constrained dynamical systems in the language of symplectic geometry. Next came a careful description, using the tools of infinite-dimensional analysis, of the space of (irreducible) connections of appropriate Sobolev class as an infinite-dimensional principal bundle, and a proof that this bundle is in general not trivial. (In this last result we were anticipated by I. M. Singer.)

We discovered that the 'Coulomb gauge condition' could be thought of as defining a connection – which we dubbed the Coulomb connection – on the infinite-dimensional principal bundle of connections. Quite remarkably, in contrast to the abelian case when this connection has zero curvature and thus yields an actual gauge-fixing, the Coulomb connection is 'maximally non-integrable' in the non-abelian case.

There were other insights, including about 'anomalies', that I was too timid to pursue.

In the late eighties, the focus of theoretical physics shifted again, as it is wont to every ten years or so. It was discovered that conformal field theory (especially in the context of the Wess-Zumino-Witten (WZW) model) has much to with line bundles on the moduli spaces of vector bundles on curves. (In fact, this turned out to involve 'parabolic bundles', a construct due to V. B. Mehta and Seshadri). These issues brought Narasimhan back to the subject. This was in important joint papers with J.-M. Drezet, S. Kumar and Ramanathan and the present author.

Teacher and guide

Narasimhan's success as a mentor and guide is legendary. He played a crucial role in the formation and development of schools in algebraic geometry, differential geometry and Lie groups at the Tata Institute. Among his students were Ramanan, M. S. Raghunathan, V. K. Patodi and Parthasarathy.

There are many traits that go into making a successful mathematician. Energy, perseverance, scholarship, clarity of thought and a mix of other less tangible ones. Narasimhan has all these in abundance, but one does not feel overwhelmed in discussions with him, because he approaches each problem from the ground up, with delicacy and determination. He seeks to understand the situation and to answer natural questions; but once he catches scent of a deep mathematical truth, is tireless in its pursuit.

His readiness to engage with younger colleagues led to extraordinary collaborations in India and elsewhere, wherein he was simultaneously teacher, mentor and co-worker. In particular, Narasimhan has been seriously engaged with the geometry community in China, and a number of young mathematicians whom he mentored at ICTP are now leading geometers there.

Not too long ago, Narasimhan happily recounted a bit of mathematical consultancy that he did from a hospital bed, which resulted in an acknowledgement for help with 'the procedure utilized to generate all possible distinct two-dimensional configurations of a given size and geometry, used for our density functional theory calculations' in a paper (Marathe, M., az-Ortiz, A. D. and Narasimhan, S., *Ab initio* and cluster expansion study of surface alloys of Fe and Au on Ru(0001) and Mo(110): Importance of magnetism. *Phys. Rev. B*, 2013, **88**, 245442.) with a rather unlikely title.

Creating structures for promoting mathematics

Narasimhan takes administration seriously. He inspires devotion among administrative staff, who find him clear, prompt, correct and helpful. Even during periods of intense research, I have seen him move seamlessly from mathematics to correcting the draft of a letter and back again to the blackboard. As Dean of Mathematics and a senior member of the faculty at TIFR, later as Head of the Mathematics Group in ICTP, and as member of important committees, he relished dealing with the challenges of managing real-world issues. I cannot do better than quote Narasimhan's own words (in his interview with Sujatha):

'I was always interested in creating structures for promoting mathematical research in India and in developing countries. In India as Chairman of the National Board of Higher Mathematics and internationally as member of EC of IMU and President of IMU's Commission on Development of Exchange, I had some experience in this direction....

I enjoyed my work and career both in India and abroad; I had ample support from institutions in India and abroad for carrying out my personal research and for working for the development of mathematics. Working abroad at ICTP gave an opportunity to interact with young mathematicians from all over the world and help them in furthering their research. This, like my role in TIFR, gave me immense satisfaction.'

Scholar and bibliophile

The word 'bibliophile' conjures up a picture of someone obsessed with dusty first editions, rather more interested in the condition of a book than its contents. As

such, it would not apply to Narasimhan, but it will have to do for want of a better term.

Apart from Carnatic and Western classical music, books – detective fiction in particular – and serious broadsheets are Narasimhan's main diversion. During his years in Trieste, he used to travel downtown to the train terminus to pick up Sunday supplements of the Italian financial paper *Il Sole 24 Ore* (which has a good cultural and literary section), *The Times Literary Supplement, The New York Review of Books, Le Monde Diplomatique* and *The Guardian Weekly*. (His politics is decidedly left-wing.)

Tamil literature is a passion. He stocks up during his visits to Chennai, sometimes timing his trips to catch the Chennai Book Fair.

He has one of the most extensive collection of mathematical books – treatises, collected works and old classic texts – that I have seen in private hands. The libraries at TIFR and ICTP owe their excellence in large part to his stewardship over decades. When Narasimhan is at ICTP, he is a familiar figure in the library, looking over the latest journals and books or renewing his acquaintance with the old masters. Particular favourites are Poincaré and Hermann Weyl.

Let me use this opportunity to make widely known some sets of notes by Narasimhan. These are gems of mathematical exposition. Items (2)–(4) below are available on-line; the others are not, but are well worth tracking down.

(1) The standard conjectures on algebraic cycles. In *Perspectives in Mathematical Sciences* II, World Scientific, Singapore, 2009.
(2) Borel–Weil theorem – notes of lectures at the college on representation theory of Lie groups, ICTP, Trieste, 1985 (http://cdsagenda5.ictp.it/full_display.php?ida=a02103).
(3) Differential geometry – notes of lectures at the Summer School on Fibre Bundles and Geometry, ICTP, Trieste, 1982 (http://cdsagenda5.ictp.it/full_display.php?ida=a02335).
(4) Deformations of complex structures and holomorphic vector bundles. In Proceedings of the Summer School on Complex Analysis, ICTP, Trieste, 1980; *Lecture Notes in Mathematics*, 1982, **950**.
(5) Lectures on partial differential equations – course of lectures given at

the TIFR Centre for Applied Mathematics, Bangalore, 1979.
(6) Vector bundles on compact Riemann surfaces. In Proceedings of the International Seminar Course on Complex Analysis and its Applications, Trieste (1975), International Atomic Energy Agency, 1976.

Aphorisms and advice

Narasimhan is rather spare with counsel. Over the years, though, some pieces of advice have stuck in my mind. These are worth repeating because of the weight of practical wisdom they carry. I omit quotes because I am not reproducing Narasimhan's words verbatim:

(1) (While learning new mathematics) do not spend too much effort on hard exercises; save your energy for research problems.
(2) (While studying or doing research) understand simple cases first.
(3) Learn and think about any piece of mathematics from the most advanced/sophisticated point of view you are capable of.
(4) Administration is important and deserves application of thought.
(5) While replying to an important letter or e-mail, sit on the draft for a day.
(6) (As educator/researcher/administrator) you have been helped by those who went before, so you should help those who come after.
(7) At the beginning of the working day, sit in your office and do some relatively concrete mathematics – say, compute a homotopy group – even if your work is going nowhere. If you are lucky enough to be in the midst of a project with its own momentum, contrive to end each day with a concrete task programmed for the next day.

Conclusion, thanks and best wishes

I close with the text accompanying the announcement of the King Faisal International Prize for Science (2006): '...Narasimhan's work is primarily in algebraic geometry, particularly the theory of holomorphic vector bundles on compact Riemann surfaces. However, over the past 35 years, his work covered nearly all other aspects of mathematics,

while maintaining its high originality and impeccable taste, and links with the works of the greatest mathematicians.

Narasimhan's brilliant career as a mathematician and educator has taken him to major universities and institutions worldwide, and has won him the admiration of the entire community of mathematicians.'

I am sure all of Narasimhan's friends, collaborators and students, and those who have benefitted from his ideas and initiatives, join me in expressing deep gratitude and best wishes.

Appendix 1

A *manifold* of dimension n is a space whose points are locally labelled by n real coordinates – when two different sets of coordinates can be used, they will be related by a differentiable coordinate change. An elementary example is the sphere $\mathcal{S}^2$ of unit vectors in three-space. Associated to each point of a manifold is the *tangent space* at that point. In the case of $\mathcal{S}^2$, this is the tangent plane, a two-dimensional vector space. Even though the tangent spaces at any two different points of an n-dimensional manifold are isomorphic as vector spaces (both being n-dimensional vector spaces), in general they are not *naturally* isomorphic. So there is no natural way to compare tangent vectors at different points. The solution is to consider the different vector spaces 'together but separately', and introduce the notion of *tangent bundle*. This is an example of a mathematical object that we will repeatedly encounter below, namely a vector bundle.

The study of vector bundles can be regarded as linear algebra with parameters.

A *vector bundle* is a family of vector spaces (all of the same dimension, the *rank* of the bundle) parametrized smoothly – in an appropriate sense – by points of a manifold (the *base space* of the bundle). To gain some intuition, let us try to picture a vector bundle of rank 1, often called a *line bundle*. Think of a broom – a collection of midribs of coconut leaves bound together by a string. Now make an abstraction in which every rib is infinitely extended and thought of as a one-dimensional vector space, each with its own origin (zero vector) – say, the point closest to the binding string – and its own rule of vector addition. Make

a further abstraction – and this might be a stretch for the non-mathematical reader – in which each strand of the broom is labelled. Then the set of labels is the base space of the bundle.

A length function on each vector space in the family is a *metric*, and a *connection* is a way of differentiating a path of vectors in the vector bundle. Prescribing a connection gives a rule for parallel transporting vectors along any path; the derivative of the path of vectors should be zero. A *section* of the bundle is a choice of a vector in each fibre. In the case of the tangent bundle, sections are called vector fields, or more evocatively, velocity fields. If the vector bundle is a trivial line bundle – that is, if each fibre is the set of real numbers (thought of as a one-dimensional vector space) – a section is a just a real-valued function defined on the base space.

A *complex* vector bundle is one with each fibre a complex vector space. If each fibre is endowed with a hermitian inner product, we call it a *unitary* (in slightly more modern terms *hermitian*) bundle.

The first major theme of Narasimhan's work is that of 'global analysis', that is, calculus on manifolds – in particular, partial differential operators acting between vector bundles.

In coordinates, this means the study of systems of linear partial differential equations. On a manifold, the topology (global shape) imposes restrictions on solutions, restrictions that are summarized in terms of cohomology groups. (For example, a curl-free vector field on a simply connected domain in three-space is the gradient of a function.)

In the theory of analytic (=holomorphic) functions of a complex variable $z = x + iy$, we are familiar with the rich phenomenon engendered by the Cauchy–Riemann equations $\frac{\partial}{\partial \bar{z}} \phi = 0$, where

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left\{ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right\}.$$

This is the archetypical elliptic operator. To transplant this from domains in the complex plane to a real two-dimensional manifold, we need to endow the manifold with a 'complex structure' – namely local coordinates whose mutual dependence is holomorphic. In complex function theory, such 'one-dimensional complex manifolds' arise as Riemann

surfaces associated to multi-valued functions. Freed of those origins, we are free to consider compact Riemann surfaces.

We give an informal definition, which illustrates the modern, post-Bourbaki approach to mathematics. A compact Riemann surface is a compact two-dimensional manifold (a 'surface'), together with an additional *complex* structure. This structure can be described in many ways, but a particularly suggestive one is to say that we single out, among all infinitely differentiable complex-valued functions, a subset (in fact, a subalgebra) which we declare to be holomorphic. This set has to be rich enough to include local coordinates around each point, and the local coordinates have to be mutually analytic. Riemann surfaces have a number of aspects, two of which we list below:

(1) Topologically, such a surface is classified as a sphere, torus (the inner tube of a tyre), or a number – this number is the *genus*, traditionally denoted g – of such tori glued together. Of these, the sphere carries a unique complex structure – in this *avatar* it is called the Riemann sphere. Starting with genus, the complex structure allows inequivalent deformations. (More of this below.)

(2) A compact Riemann surface has a unique *algebraic structure*. This is uncovered by considering the field of meromorphic functions. (Meromorphic functions are functions analytic outside finitely many points where they are allowed to have poles, e.g. $z \mapsto z^{-n}$ with n any positive integer – but not essential singularities – e.g. $z \mapsto \exp\ z^{-1}$). In general, the field of meromorphic functions has transcendence degree one over the complex numbers, and quite remarkably, determines the complex geometry of the surface. In case of the Riemann sphere, this is the field of rational 'functions' (i.e. ratios of polynomials) in one variable. As a complex object, a Riemann surface has dimension one, so Riemann surface = complex curve.

Before we go further, an interlude to introduce the vast subject of algebraic geometry. The theory of one-dimensional complex manifolds (Riemann surfaces) can be generalized to arbitrary dimen-

sions, and this leads to the notion of a complex manifold. In higher dimensions, not every complex manifold is algebraic, and those which do admit an algebraic structure – *complex algebraic varieties* – occupy a central place in modern mathematics. An *affine* complex algebraic variety is the set of solutions of a collection of algebraic equations in finitely many (complex) variables; glueing such varieties yields more general varieties, in particular *projective* varieties, which are closed subvarieties of a complex projective space. If the equations have *integer* coefficients, they can be read modulo any prime number p, and we can also *count* the number of solutions in finite extensions of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. (These numbers will be finite if the corresponding complex variety is projective.) The famous Weil conjectures, proved by P. Deligne, relate the topology of a complex projective variety to the numbers of such solutions, provided that both the complex variety and the variety defined over $\mathbb{F}_p$ are nonsingular. (In the latter case, 'nonsingular' must be suitably defined.)

Returning to Riemann surfaces, we have the following remarkable picture. Fix a genus g, and consider the set of complex (=algebraic) structures on an (oriented) surface of genus g. This set of complex structures has itself a natural structure of a complex algebraic variety $\mathcal{M}_g$ of dimension $3g - 3$. This variety is a basic example of a *moduli space*. A point of $\mathcal{M}_g$ determines an algebraic structure on the surface, and as we move away from this point, the algebraic structure changes or 'deforms'. Historically, the local parameters were called moduli; hence the term 'moduli space'.

It is worth elaborating on the notion of deformation. If we consider a fixed topological space, it may or may not admit a structure of differentiable manifold. Such a structure, if it exists, need not be unique; nonetheless, any differentiable structure 'close enough' to a given one will be isomorphic to it. This is no longer the case for complex structures, and in fact, the space of complex structures on a given differentiable manifold is explored by fixing one complex structure and probing nearby ones in 'perturbation theory'. The machinery goes by the name of deformation theory.

Let us now consider complex vector bundles. On a Riemann surface, such a bundle is topologically classified by its

rank (= the dimension of each fibre) and *degree* or *first Chern class* (an integer that describes how 'twisted' the bundle is). Recall the notion of a section of a vector bundle – a choice of a vector in each fibre. A *holomorphic structure* on a complex vector bundle is a choice of a subset of sections which we declare to be holomorphic. As in the case of complex structures on a surface, the space of holomorphic sections has to be 'big' enough – multiplying a holomorphic section by a holomorphic function should result in a new holomorphic section, and values of holomorphic sections should locally span the fibres of the vector bundle. In general, a given complex vector bundle admits inequivalent holomorphic structures. In fact, the set of holomorphic structures on the trivial line bundle is already an interesting moduli space, the Jacobian of the Riemann surface, which is a complex torus of dimension g.

When we turn to vector bundles of rank bigger than one, it turns out that there is no way to put together the isomorphism classes of *all* holomorphic vector bundles into a variety. On the other hand, if we consider *semistable bundles*, there is indeed a nice variety

that parametrizes (strictly speaking not isomorphism classes but *s-equivalence classes* of) semistable bundles. Fixing rank r and degree d, this is the variety $U_X(r, d)$ referred to in the main body of this article.

Appendix 2

Lemma 1. *Let a real C^∞ one-form $B = \lambda_1(x)\mathrm{d}x_1 + \lambda_2(x)\mathrm{d}x_2$ be given in an open ball V in $\mathbb{R}^2$. Then we can find (possibly in a smaller ball) C^∞ complex-valued functions u_l, $l = 1,\ldots, 5$ such that*

$$\sum_l |u_l(x)|^2 = 1, \text{ and} \tag{1}$$

$$\sum_l \bar{u}_l(x)\mathrm{d}u_l = iB. \tag{2}$$

(Note that the first equation implies that the form on the left of the second equation is purely imaginary.) The following proof is simple, but tricky.

Proof. At the cost of shrinking the ball and scaling the coordinates, we can assume that

$$\lambda_j = \mu_j - \nu_j, j = 1, 2,$$

with (a) both μ_j and ν_j strictly positive functions; (b) $\mu_j \ll 1$, $\nu_j \ll 1$, and (c) $\sum_j \mu_j + \nu_j < \frac{1}{2}$. Define functions p_j and q_j by $p_j^2 = \mu_j$, $q_j^2 = \nu_j$. Set $u_j = p_j e^{+ix_j}$, $u_{2+j} = q_j e^{-ix_j}$, and define the (real-valued) positive function u_5 by

$$u_5^2 = 1 - \sum_j (p_j^2 + q_j^2),$$

Clearly we have $\sum_l |u_l(x)|^2 = 1$. Also

$$\sum_l \bar{u}_l(x)\mathrm{d}u_l = i\sum_j (p_j^2 - q_j^2)\mathrm{d}x_j$$
$$+ \sum_j (p_j\mathrm{d}p_j + q_j\mathrm{d}q_j) + u_5\mathrm{d}u_5 = iB.$$

T. R. RAMADAS

Chennai Mathematical Institute,
H1 Sipcot IT Park,
Siruseri 603 103, India
e-mail: ramadas@cmi.ac.in