Fully Coherent Slow Frequency-hop Spread Spectrum

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In course of a study on spread spectrum (S/S) systems and their uses, a discussion of coherent slow frequency-hop spread-spectrum (FH S/S) systems presented in Ziemer & Peterson led the present author to search for a way to derive the power spectral density (PSD) function for such a waveform. The result presented in the quoted reference was acknowledged to a collection of papers by R. C. Titsworth & L. R. Welch, entitled, "Power Spectra of Signals Modulated by Random & Pseudorandom Signals". The original derivation not being available, the present author has endeavoured to present an independent one, maintaining that it is unknown whether the latter is different from the former or not. The validity of this process is borne out by the result being identical. To this it may be added that the procedure was ratified by Prof. A. K. Dutta, currently at the Centre for Advanced Studies, Science College, Calcutta University.

Before presenting the derivation, an overview of slow FH S/S is in place. Frequency-hop spread spectrum systems evolved out of the need to foil jammer strategies more successfully. It was envisaged to randomly hop the data-modulated carrier from one frequency to the next. The spectrum of the transmitted signal is spread sequentially, using pseudorandom ordered sequence of hops, rather than instantaneously. A common modulation format for FH systems is that of M-ary Frequency Shift Keying (MFSK), so that the combination is referred to as FH/MFSK. Based on the rate at which the hops occur, there are two basic (technology independent) characterizations of FH S/S :

1. Slow FH : Symbol-rate of MFSK signal is an integral multiple of the hop rate. Several symbols are transmitted on each frequency hop.
2. Fast FH : Hop rate an integral multiple of symbol rate. Carrier frequency changes or hops several times during the transmission of a single symbol.

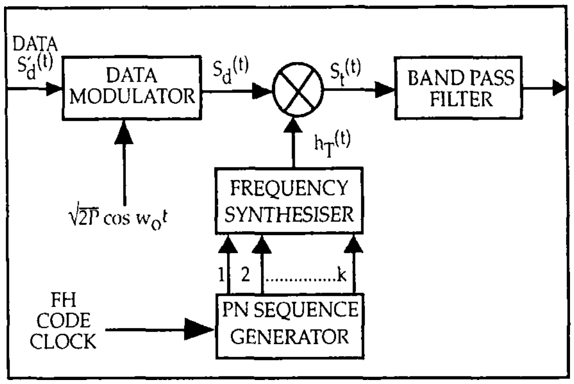

Fig. - 1

The fully coherent slow FH/SS scheme is shown in the figure. Frequency synthesiser output is a sequence of tones, each of duration Tc, so that

$$h_T(t) = \sum_{n=-\infty}^{\infty} 2p\,(t-nT_c)\cos(\omega_n t + \varnothing_n)$$

where $p(t) = \begin{cases} 1, & o \le t \le T_c \\ o, & \text{elsewhere} \end{cases}$

and ω_n and $\varnothing_n$ are the angular frequency and phase associated with the n-th frequency hop interval. The frequency ω_n is chosen from a set of 2^k frequencies. The transmitted signal is the data modulated carrier up-converted to a new frequency $(\omega_o+\omega_n)$ on each FH chip.

$$\text{So, } s_t(t) = \left[s_d(t)\sum_{n=-\infty}^{\infty} 2p\,(t-nT_c)\cos(\omega_n t+\varnothing_n)\right]_{\text{sum frequency components}}$$

The transmitted power spectrum is given by

$[S_d(f) \star S_h(f)]_{\text{sum frequency components}}$

Here $S_d(f)$ is the power spectral density (PSD) of the data modulated carrier and $S_h(f)$ is the PSD of the hop-carrier $h_T(t)$. Assuming independence of the two sets of carriers, the power spectrum is the sum frequency component of the convolution.

Now we address the problem of evaluating the PSD of the hop carrier, $h_T(t)$. The signal $h_T(t)$ may or may not be periodic in nature.

Even if it is periodic, its period is long enough to be considered infinite with little error. So $h_T(t)$ is assumed a purely random sequence of frequencies. If this be the case, we assume further that $h_T(t)$ is a sample function of a wide-sense stationary random process. From the pulse sequence representation we also get that $h_T(t)$ being complex is intuitively satisfying. Dispensing with Markov sources, which would have made $h_T(t)$ too troublesome to handle, we propose the bolder simplification of ergodicity. Our contention stems from the fact that the defining pulse shape for $h_T(t)$ remains the same over all interval $[kT_c, (k+1)T_c]$, k being integral, as $p(t) = \begin{cases} 1, & o \le t \le T_c \\ 0, & \text{otherwise} \end{cases}$

Then $Rh_T(\tau) = E[h_T(t+\tau)\, h_T^*(t)$ by definition.

This will equal the expectation of $[h_T(t+\tau)\, h_T^*(t)]$ over some time interval $0 \le t \le T_c$, i.e. $h_T(t)$ is assumed ergodic in autocorrelation.

Let $Rh(\tau) = \{E[R_m](\tau)\}$ where the expectation operator takes R_m as its argument and not τ.

Accordingly, we define $R_m(\tau) = \frac{1}{Tc}\int_0^{T_c} g_m(t+\tau)\, g_m^*(t)\, dt$

where $g_m(t) = \begin{cases} 2p(t)\cos(\omega_m(t)+ø_m), & o \le t \le T_c \\ 0, & \text{elsewhere} \end{cases}$

Taking Fourier transform

$$S_h(f) = \frac{1}{T_c}\; E\left[\int_{-\infty}^{\infty}\{(g_m(t))^*\,(g_m^*(t)\}\exp(-j2\pi\, ft)dt\right]$$

$$= \frac{1}{T_c} E[G_m(f)\, G_m^*(f)],$$ interchanging the expectation and Fourier transform operators.

$G_m(f)$ is composed of a summation of m functions with specific probabilities and with their particular frequency content. This representation is analogous to expression of a particular signal in terms of m (non) orthogonal signal basis vectors with proper coefficients. In the specific case, when $G_m(f)$ has two components $G_{m1}(f)$ and $G_{m2}(f)$ with associated probabilities P_1 and P_2 we have

$$E[G_m(f)\, G_m^*(f)] = \sum_{m=1}^{2} G_m(f)\, p_m \sum_{m'=1}^{2} G_{m'}(f)\, p_{m'}$$

$$= p_1^2|G_1(f)|^2 + P_2^2|G_2(f)|^2 + p_1p_2G_1(f)G_2^*(f) + p_1p_2G_1^*(f)G_2(f)$$

$$= p_1^2|G_1(f)|^2 + P_2^2|G_2(f)|^2 + p_1p_2\{|G_1(f)|^2 + |G_2(f)|^2 - 2Re[G_1(f)G_2^*(f)]\}$$

$$= p_1|G_1(f)|^2 + P_2|G_2(f)|^2 - 2p_1p_2 Re[G_1(f)G_2^*(f)] \text{ as } p_1+p_2 = 1$$

Directly applying this result when there are 2^k components of

$$G_m(f), E[G_m(f)G_m^*(f)] = \sum_{m=1}^{2^k} p_m|G_m(f)|2 - 2\sum_{m=1}^{2^k}\sum_{\substack{m'=1\\ m'\neq m\\ m'>m}}^{2^k} p_m p_m\, re[G_m(f)G_m^*(f)]$$

But there is a periodic component in $h_T(t)$ in that it attains the same pahse $ø_m$ whenever the frequency ω_m is selected. The quantity $R_h(\tau)$ has to be modified so that its transform has some discrete frequency components. These periodic components may be represented by $\sum_{k=-\infty}^{\infty} R_{gmm}(\tau - kT_c)$ and the probability associated with each such term is $p_m\, p_m = p_m^2$. This is because cross correlation among $g_m(t)$ and $g_{m'}(t)$ when the shift $|\tau| = kT_c$, k = 1,2,3, ..., is zero.

Taking Fourier transform, for a given m,

$$F\left[\sum_{k=-\infty}^{\infty} R_{gmm}(\tau - kT_c)\right] = \sum_{k=\infty}^{\infty} |G_m(k/T_c)|^2\, \partial(f - {}^k/_{Tc})$$

So the final expression for the PSD is :

$$Sh(f) = \frac{1}{T_c^2}\sum_{n=-\infty}^{\infty}\left|\sum_{m=1}^{2^k} P_m^2\, G_m({}^n/_{Tc}\right|^2 \partial(f - {}^n/_{Tc})$$

$$+ \frac{1}{T_c}\sum_{m=1}^{2^k} P_m(1-P_m)\,|G_m(f)|^2 - \frac{2}{T_c}\sum_{m=1}^{2^k}\sum_{\substack{m'=1\\ m'\neq m\\ m'>m}}^{2^k} P_m P_{m'}\, Re[G_m(f)G_{m'}^*(f)]$$

$$\text{where } G_m(f) = F\left[\text{rect}\left(\frac{t - {}^{Tc}/_2}{T_c}\right)\{\exp(j\omega_m t + jø_m) + \exp(j\text{-}j\omega_m t - jø_m)\}\right]$$

$$= T_c\exp\{-j[\pi(f-f_m)T_c - ø_m]\}\,\text{sine}[(f-f_m)T_c]$$

$$+ T_c\exp\{-j[\pi(f+f_m)T_c + ø_m]\}\,\text{sine}[(f+f_m)T_c\}$$

The derivation is now complete. But certain working assumptions need to be made to dispel its cumbersome aspect. In this, and subsequent developments, Ziemer and Peterson have been followed.

When T_c is suitably large, i.e. hop rate $1/T_c$ quite small with respect to maximum frequency spacing, $G_m(f)$ and $G_m(f)$ are nearly non-overlapping (orthogonal) and hence the overlap term is zero. Also assuming all frequencies equiprobable, the modified PSD is :

$$S_h(f) \approx \frac{1}{2^{2k}} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{2^k} \mathrm{sinc}^2 [(n - f_m T_c)] + \mathrm{sinc}^2 [(n + f_m T_c)]\} \times \partial(f - {}^n/_{T_c})$$

$$+ \frac{T_c}{2^k} \left(1 - \frac{1}{2^k}\right) \sum_{m=1}^{2^k} \{ \mathrm{sinc}^2 [(f - f_m) T_c] + \mathrm{sinc}^2 [(f + f_m) T_c]\}$$

If minimum frequency spacing is so chosen to be an integral multiple of hop rate, $f_m T_c$ is an integer, and the sinc function of the first term sampled only at integer values. So,

$$S_h \approx \frac{1}{2^{2k}} \sum_{m=1}^{2^k} [\partial (f - f_m) + \partial(f + f_m)]$$

$$+ \frac{T_c}{2^k} \left(1 - \frac{1}{2^k}\right) \sum_{m=1}^{2^k} \{ \mathrm{sinc}^2[(f - f_m) T_c] + \mathrm{sinc}^2 [f + f_m)T_c]$$

If the signal s_d (t) be the result of BPSK data modulation, its PSD is : $S_d(f) = {}^{PT}/_2 \{ \mathrm{sinc}^2 [(f - f_0) T]\} + \mathrm{sinc}^2 [(f + f_0) T]\}$ PSD of the total signal st (t) obtained by frequency convolution observing that only the sum frequency components of the result need be considered. As in slow FH systems, data rate> hop rate, one of the sinc functions in the convolution, pertaining to the hop PSD, may be approximated as a constant with respect to the other. The frequency convolution thus, acquires more and more the nature of a frequency translation of the function exhibiting quicker variations. The final approximate result is, therefore.

$$S_t(f) \approx \frac{PT}{2.2^k} \sum_{m=1}^{2^k} \{\mathrm{sinc}^2 [(f - f_m - f_0) T] + \mathrm{sinc}^2 [(f + f_m + f_0)T]\}$$

$$+ \frac{PT}{2.2^k}(1 - 2^{-k}) \{\mathrm{sinc}^2 [(f - f_m - f_0) T] + \mathrm{sinc}^2 [(f + f_m + f_0)T]\}$$

$$\text{Or, } S_t(f) \approx \frac{PT}{2^{k+1}} \sum_{m=1}^{2^k} \{\mathrm{sinc}^2 [(f - f_m - f_0) T] + \mathrm{sinc}^2 [(f + f_m + f_0)T]\}$$

The discrete components of $S_h(f)$ are negligible only when $T_c 2^k >> 1$, which, however, is not generally the case.

Coherent frequency hop systems are amenable to analysis and provides both practical and pedagogical insights of the spectral composition of FH S/S signals. Non-coherent systems pose many difficulties in this respect. But the former has serious problems pertaining to reception.

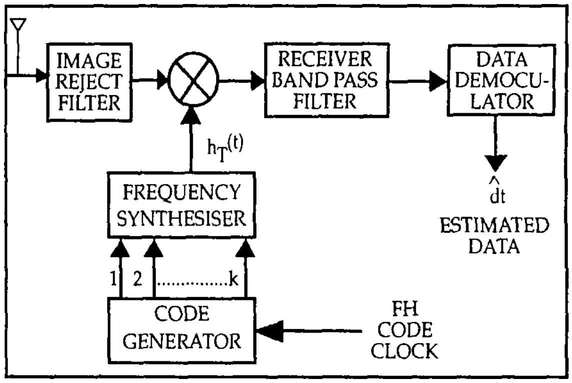

Figure 2

In the figure, the receiver gets a signal $s_t(t-T_d)$ given by : $s_t(t-T_d) = \sqrt{2P} \sum_{n=-\infty}^{\infty} p(t - T_d - nT_c) \cos [(\omega_0 + \omega_n) t + ø_n + \theta d (t - T_d) - (\omega_0 + \omega_n) T_d]$ The signal is down converted by a locally generated reference $h_R(t) = 2 \sum_{n=-\infty}^{\infty} p(t - \hat{T}_d - nT_c) \cos [\omega_n t + øn - \omega_n \hat{T}_d]$

Bandpass filtering extracts difference frequency components of down connersion mixer. Assuming $\hat{T}_d = T_d$

$$y(t) = [s_t(t - T_d) h_R(t)]_{LP} \sqrt{2p} \sum_{n=-\infty}^{\infty} p^2 (t - T_d - nT_c) \cos [(\omega_0 + \omega_0 Td + \theta d(t - Td)]$$

$$= \sqrt{2p} \cos [\omega_0 (t - T_d) + \theta_d (t - T_d)]$$

and data modulated carrier recovery is complete. But in the case $T_d \neq \hat{T}_d$, recovered carrier is phase modulated by terms of the forms $\sum_n (T_d - \hat{T}_d) \omega_n$. This problem cannot be remedied unless a means is provided for coherent carrier tracking indipendent of the FH code tracking loop.

Concluding, the rather involved expression for PSD of coherent slow FH S/S is of more than just pedagogical interest in that it gives a 'feel' of the frequency composition of a slow FH S/S. This is important whenever one has to take stock of an GH S/S situation without having either the time or the means to analyse it experimentally. The author would like to end on a note of gratitude to Prof A. K. Dutta for his inspirational guidance.

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FROM MINUTES TO ETERNITY

Little drops of water ...
Little grains of sand ...

Make the mighty ocean
and the pleasant land

So, the little minutes
Humble though they be...
make the mighty ages of

E T E R N I T Y !